1. CLAIM: $x, y \in O \Rightarrow x+y \in E$

PROOF: SUPPOSE $x, y \in O$

$x \in O$, I.E., $\exists\, a \in \mathbb{Z}$ WITH $x = 2a+1$

$y \in O$, I.E., $\exists\, b \in \mathbb{Z}$ WITH $y = 2b+1$

THEN $x+y = (2a+1)+(2b+1)$
$= 2a+2b+2$ (*)

SCRATCH WORK: k?
WANT $2a+2b+2 = 2k$
$\therefore$ $k = a+b+1$

TAKE $k = a+b+1 \in \mathbb{Z}$.

THEN BY DEF'N, $E \ni 2k = 2(a+b+1) = 2a+2b+2 \overset{*}{=} x+y$,
SO $x+y \in E$. ■

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2. CLAIM: $x \in E \wedge y \in O \Rightarrow xy \in E$.

PROOF: SUPPOSE $x \in E$, I.E., $\exists\, a \in \mathbb{Z}$ WITH $x = 2a$
AND $y \in O$, I.E., $\exists\, b \in \mathbb{Z}$ WITH $y = 2b+1$.

THEN $xy = 2a(2b+1) = 4ab+2a$ (*)

SCRATCH WORK: k?
WANT $4ab+2a = 2k$
$\therefore$ $k = 2ab+a$

TAKE $k = 2ab+a \in \mathbb{Z}$.

THEN BY DEF'N, $E \ni 2k = 2(2ab+a) = 4ab+2a = xy$,
SO $xy \in E$. ■

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3. CLAIM: $x \in O \Rightarrow \frac{x+1}{2} \in \mathbb{Z}$

PROOF: SUPPOSE $x \in O$, I.E., $\exists\, a \in \mathbb{Z}$ WITH $x = 2a+1$

THEN $\frac{x+1}{2} = \frac{(2a+1)+1}{2} = \frac{2a+2}{2} = a+1 \in \mathbb{Z}$, SO $\frac{x+1}{2} \in \mathbb{Z}$. ■

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5. CLAIM: $U \subset O$ (I.E., $x \in U \Rightarrow x \in O$)

PROOF: SUPPOSE $x \in U$, I.E., $x = 6a+1$ FOR SOME $a \in \mathbb{Z}$.

SCRATCH WORK: k?
WANT $6a+1 = 2k+1$
$\therefore$ $k = 3a$

TAKE $k = 3a \in \mathbb{Z}$.

THEN BY DEF'N,

$O \ni 2k+1 = 2(3a)+1 = 6a+1 = x$, SO $x \in O$. ■

6. CLAIM: $x \in T \Rightarrow 2x-5 \in U$ $\oplus$

PROOF: SUPPOSE $x \in T$, I.E., $x = 3a$ FOR SOME $a \in \mathbb{Z}$.

THEN $2x-5 = 2(3a)-5$
$= 6a-5$ (*)

SCRATCH WORK: k?
WANT $6a-5 = 6k+1$
$\therefore 6a-6 = 6k$
$\therefore k = a-1$

TAKE $k = a-1 \in \mathbb{Z}$.

THEN BY DEF'N, $U \ni 6k+1 = 6(a-1)+1 = 6a-5 \overset{*}{=} 2x-5$,
SO $2x-5 \in U$. ∎

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7. CLAIM: $x \in Q \Rightarrow x^2 \in U$ $\oplus$

PROOF: SUPPOSE $x \in Q$, I.E, $x = 6a+5$ FOR SOME $a \in \mathbb{Z}$

THEN $x^2 = (6a+5)^2$
$= 36a^2+60a+25$ (*)

SCRATCH WORK: k?
WANT $36a^2+60a+25 = 6k+1$
$\therefore 36a^2+60a+24 = 6k$
$\therefore k = 6a^2+10a+4$

TAKE $k = 6a^2+10a+4 \in \mathbb{Z}$.

THEN BY DEF'N, $U \ni 6k+1 = 6(6a^2+10a+4)+1$
$= 36a^2+60a+25 \overset{*}{=} x^2$,
SO $x^2 \in U$. ∎

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8. CLAIM: $x \in Q \wedge y \in U \Rightarrow x-y \in E$ $\oplus$

PROOF: SUPPOSE $x \in Q$, I.E., $x = 6a+5$ FOR SOME $a \in \mathbb{Z}$
AND $y \in U$, I.E., $y = 6b+1$ FOR SOME $b \in \mathbb{Z}$

THEN $x-y = (6a+5)-(6b+1)$
$= 6a-6b+4$ (*)

SCRATCH WORK: k?
WANT $6a-6b+4 = 2k$
$\therefore k = 3a-3b+2$

TAKE $k = 3a-3b+2 \in \mathbb{Z}$.

THEN BY DEF'N, $E \ni 2k = 2(3a-3b+2) = 6a-6b+4 \overset{*}{=} x-y$,
SO $x-y \in E$. ∎

10. CLAIM: $x \in Q \wedge y \in U \Rightarrow x+y \in E \cap T$

$\oplus$ I.E., $x+y \in E \wedge x+y \in T$

AVOID USING k AGAIN!

PROOF: SUPPOSE $x \in Q$, I.E., $x = 6a+5$ FOR SOME $a \in \mathbb{Z}$
AND $y \in U$, I.E., $y = 6b+1$ FOR SOME $b \in \mathbb{Z}$

THEN $x+y = (6a+5)+(6b+1)$
$= 6a+6b+6$ $(*)$

SCRATCH WORK: k?
WANT $6a+6b+6 = 2k$
$\therefore k = 3a+3b+3$

TAKE $k = 3a+3b+3 \in \mathbb{Z}$.

THEN BY DEF'N, $E \ni 2k = 2(3a+3b+3) = 6a+6b+6 = x+y$,
SO $x+y \in E$.

SCRATCH WORK: ℓ?
WANT $6a+6b+6 = 3\ell$
$\therefore \ell = 2a+2b+2$

NOW TAKE $\ell = 2a+2b+2 \in \mathbb{Z}$.

THEN BY DEF'N,
$T \ni 3\ell = 3(2a+2b+2) = 6a+6b+6 \overset{*}{=} x+y$,
SO $x+y \in T$.

SINCE $x+y \in E \wedge x+y \in T$, BY DEF'N, $x+y \in E \cap T$. ■

EXTRA SKILLS PRACTICED IN #4 & #9:

- PROVING SET EQUALITY: $A = B$ MEANS $x \in A \Leftrightarrow x \in B$
- PROVING $\Leftrightarrow$ STATEMENTS: $P \Leftrightarrow Q$ MEANS $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- WORKING WITH "$\vee$" IN PROOFS:
  - WHEN SUPPOSING AN "$\vee$": SPLIT CASES
  - IN THESE EXAMPLES, TAKING REMAINDERS AFTER DIVIDING BY SOME # CAN HELP US PROVE AN "$\vee$"

4. $\mathbb{Z} = E \cup O$ MEANS $x \in \mathbb{Z} \Leftrightarrow x \in E \cup O$,

I.E., $\underbrace{(x \in \mathbb{Z} \Rightarrow x \in E \cup O)}_{①} \wedge \underbrace{(x \in E \cup O \Rightarrow x \in \mathbb{Z})}_{②}$

CLAIM ①: $x \in \mathbb{Z} \Rightarrow x \in E \cup O$ ⊕ I.E., $x \in E \vee x \in O$

PROOF ①: SUPPOSE $x \in \mathbb{Z}$.

DIVIDE x BY 2 AND CONSIDER POSSIBLE REMAINDERS:

- 0: $x = 2q + 0$ FOR SOME $q \in \mathbb{Z}$ $\therefore$ BY DEF'N OF E, $x \in E$
- 1: $x = 2q + 1$ FOR SOME $q \in \mathbb{Z}$ $\therefore$ BY DEF'N OF O, $x \in O$

THUS, $x \in E \vee x \in O$, SO $x \in E \cup O$. ■

CLAIM ②: $x \in E \cup O \Rightarrow x \in \mathbb{Z}$ ⊕

PROOF ②: SUPPOSE $x \in E \cup O$, I.E., $x \in E \vee x \in O$.

SPLIT CASES
- IF $x \in E$: THEN $x = 2a$ FOR SOME $a \in \mathbb{Z}$ $\therefore$ $x \in \mathbb{Z}$
- IF $x \in O$: THEN $x = 2b + 1$ FOR SOME $b \in \mathbb{Z}$ $\therefore$ $x \in \mathbb{Z}$

IN EITHER CASE, $x \in \mathbb{Z}$. ■

9. $\mathbb{Z} = E \cup T \cup U \cup Q$ MEANS $x \in \mathbb{Z} \Leftrightarrow x \in E \cup T \cup U \cup Q$,

i.e., $\underbrace{(x \in \mathbb{Z} \Rightarrow x \in E \cup T \cup U \cup Q)}_{①} \wedge \underbrace{(x \in E \cup T \cup U \cup Q \Rightarrow x \in \mathbb{Z})}_{②}$

CLAIM ①: $x \in \mathbb{Z} \Rightarrow x \in E \cup T \cup U \cup Q$ $\oplus \to$ $x \in E \vee x \in T \vee x \in U \vee x \in Q$

PROOF ①: SUPPOSE $x \in \mathbb{Z}$.

PATTERN-MATCHING INDICATED WITH UNDERLINE; CAN ALSO DO FULL SCRATCH WORK TO TAKE K!

DIVIDE x BY 6 AND CONSIDER POSSIBLE REMAINDERS:

- 0: $x = 6q + 0$ FOR SOME $q \in \mathbb{Z}$, SO $x = 2(3q)$ WITH $3q \in \mathbb{Z}$ $\therefore$ $x \in E$
- 1: $x = 6q + 1$ FOR SOME $q \in \mathbb{Z}$, SO $x \in U$ BY DEF'N OF U.
- 2: $x = 6q + 2$ FOR SOME $q \in \mathbb{Z}$, SO $x = 2(3q+1)$ WITH $3q+1 \in \mathbb{Z}$ $\therefore$ $x \in E$
- 3: $x = 6q + 3$ FOR SOME $q \in \mathbb{Z}$, SO $x = 3(2q+1)$ WITH $2q+1 \in \mathbb{Z}$ $\therefore$ $x \in T$
- 4: $x = 6q + 4$ FOR SOME $q \in \mathbb{Z}$, SO $x = 2(3q+2)$ WITH $3q+2 \in \mathbb{Z}$ $\therefore$ $x \in E$
- 5: $x = 6q + 5$ FOR SOME $q \in \mathbb{Z}$, SO $x \in Q$ BY DEF'N OF Q.

THUS, $\underbrace{x \in E}_{\text{CASES } 0,2,4} \vee \underbrace{x \in T}_{\text{CASE } 3} \vee \underbrace{x \in U}_{\text{CASE } 1} \vee \underbrace{x \in Q}_{\text{CASE } 5}$,

SO BY DEFINITION, $x \in E \cup T \cup U \cup Q$. ■

CLAIM ②: $x \in E \cup T \cup U \cup Q \Rightarrow x \in \mathbb{Z}$ $\oplus$

PROOF ②: SUPPOSE $x \in E \cup T \cup U \cup Q$, i.e., $x \in E \vee x \in T \vee x \in U \vee x \in Q$.

SPLIT CASES
- IF $x \in E$: THEN $x = 2a$ FOR SOME $a \in \mathbb{Z}$ $\therefore$ $x \in \mathbb{Z}$
- IF $x \in T$: THEN $x = 3b$ FOR SOME $b \in \mathbb{Z}$ $\therefore$ $x \in \mathbb{Z}$
- IF $x \in U$: THEN $x = 6c + 1$ FOR SOME $c \in \mathbb{Z}$ $\therefore$ $x \in \mathbb{Z}$
- IF $x \in Q$: THEN $x = 6d + 5$ FOR SOME $d \in \mathbb{Z}$ $\therefore$ $x \in \mathbb{Z}$

IN EACH CASE, $x \in \mathbb{Z}$. ■